Caustic free completion of k-essence

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In collaboration with Eugeny Babichev and Alexander Vikman

Based on [arXiv:1704.03367, 1807.10281]

Unbearable lightness of the Universe

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$$\text{k-essence} = P(X, \varphi)$$

$$X \equiv (\partial_\mu \varphi)^2$$

Focus on shift-symmetric k-essence, or $P(X)$-theories.

No Universe, no Dark Matter, no Big Bang....

Possible pathologies in field theories

- Ghost instabilities. E.g., due to higher derivatives.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + ... \qquad (+---)$$

See, however, Deffayet, Mukohyama, Vikman'21

- Gradient instabilities.

$$\omega^2 = -|c_s|^2 k^2$$

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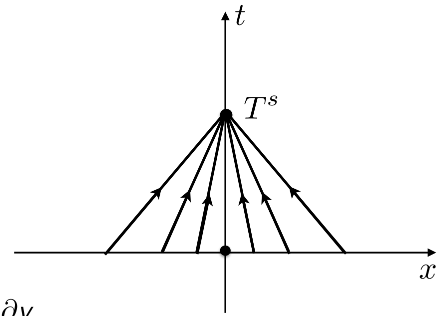
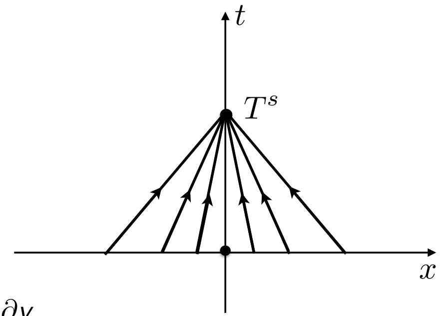
  Not always the case: $P(X)$-fluids, k-essence, Generalized Galileons

  Babichev'16.



$$\frac{\partial v}{\partial t} + (v \cdot \partial_x)\, v = 0 \qquad \text{Euler equation}$$

This simplified example: $v(t=0,x) = -\frac{x}{T_s} \Longrightarrow v(t,x) = -\frac{x}{T_s - t}$.



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Each dust particle moves along a straight line with a constant velocity
$\frac{dx}{dt} = v = \text{const}$

Crossing of trajectorities is a singularity: multi-valued velocity. Caustics!

/Caustics are formed beyond this simplified example.

Trajectories of non-interacting dust particles=characteristics of equations of motion.

Formation of caustics=characteristics of equations of motion cross.

NB. In the case of realistic particles, appearance of caustics signalizes the beginning of the multi-stream regime (shell crossing)

Lagrangian formulation of pressureless perfect fluid

$$\mathcal{L} = \frac{\lambda^2}{2}X - \frac{\lambda^2}{2} \qquad X = (\partial_\mu \varphi)^2$$

$$\Longrightarrow$$

$$T_{\mu\nu} = \rho u_\mu u_\nu \qquad \rho = \lambda^2 \qquad u_\mu = \partial_\mu \varphi \; .$$

We deal with the irrotational pressureless perfect fluid.

From pressureless perfect fluid to $P(X)$

$$\mathcal{L} = \frac{\lambda^2}{2} X - \frac{\lambda^2}{2} \qquad \rightarrow \qquad \frac{\lambda^2}{2} X - V(\lambda)$$

Lagrangian multiplier $\quad\rightarrow\quad$ Auxiliary field

$$\frac{V'(\lambda)}{\lambda} = X \Longrightarrow \mathcal{L} = P(X)$$

Pressureless perfect fluid $\quad\rightarrow\quad$ $P(X)$

Example: $\quad P(X) = X^2 \rightarrow V(\lambda) = \dfrac{\lambda^4}{4}$

Formation of caustics in $P(X)$-fluids Babichev'16

In the case of pressureless perfect fluid, along each characteristics $\frac{dx}{dt} = v$.

Generalization to $P(X)$-fluid: $\boxed{\left(\frac{dx}{dt}\right)_{\pm} = \frac{v \pm c_s}{1 \pm v c_s}}$ $\quad v \equiv -\frac{\partial\varphi}{\partial x}\left[\frac{\partial\varphi}{\partial t}\right]^{-1}$

NB: the sound speed squared $c_s^2 = \left(1 + 2X\frac{P_{XX}}{P_X}\right)^{-1}$

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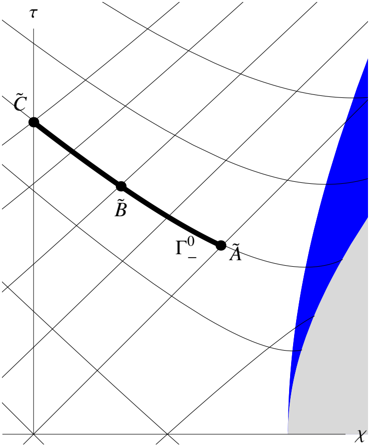
Specific cases: $P(X) = X \Longrightarrow c_s = 1 \Longrightarrow \left(\frac{dx}{dt}\right)_{\pm} = \pm 1 \Longrightarrow$ No caustics.

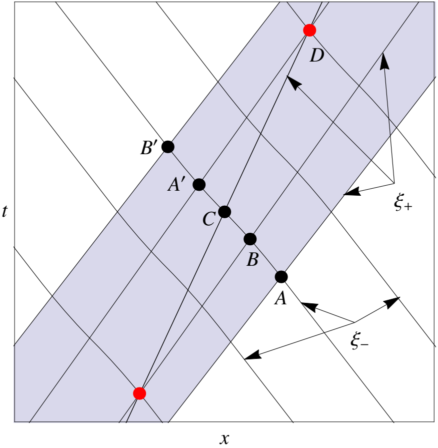
$v = \pm 1 \Longrightarrow \left(\frac{dx}{dt}\right)_{\pm} = \pm 1 \Longrightarrow$ No caustics $\qquad \frac{\partial \varphi}{\partial t} = \mp \frac{\partial \varphi}{\partial x} \Longrightarrow X = 0$

E. Babichev JHEP 1604 (2016) 129

$\tau = \frac{\partial \varphi}{\partial t}$ $\qquad \chi = \frac{\partial \varphi}{\partial x}$

Generic simple wave $\tau = \tau(\chi)$





The argument can be generalized to k-essence ($P(X,\varphi)$-fluids).

Higher derivative terms in the Generalized Galileon models do not cure caustic singularities.

First step towards curing caustic singularities: focus on P(X)-fluids.

Towards completing $P(X)$-fluid

$$\mathcal{L} = P(X) \quad \text{is equivalent to} \quad \mathcal{L} = \frac{\lambda^2}{2} X - V(\lambda) \, .$$

$$\frac{V'(\lambda)}{\lambda} = X \Longrightarrow \mathcal{L} = P(X) \, .$$

Promote an auxiliary field to a dynamical degree of freedom!

$$\mathcal{L} = \boxed{\frac{(\partial_\mu \lambda)^2}{2}} + \frac{\lambda^2}{2} X - V(\lambda)$$

$$\boxed{\Psi = \lambda e^{i\varphi} \qquad \mathcal{L} = \frac{1}{2}|\partial_\mu \Psi|^2 - V(|\Psi|)} \; .$$

Cf. Bilic'08 Bekenstein'88

The model is manifestly caustic free and exhibits luminal propagation.

$$P(X) = X^2 \Longrightarrow \frac{1}{2}|\partial_\mu \Psi|^2 - \frac{1}{4}|\Psi|^4$$

$$\text{Pressureless perfect fluid} \Longrightarrow \frac{1}{2}|\partial_\mu \Psi|^2 - \frac{1}{2}M^2|\Psi|^2$$

$P(X)$-complex field correspondence

Introducing a new degree of freedom (radial field) may be not harmless

Before caustics are formed $\quad (\partial_\mu \lambda)^2 \ll V(\lambda) \Longrightarrow$

At the time of the caustics formation $\quad (\partial_\mu \lambda)^2 \sim V(\lambda)$

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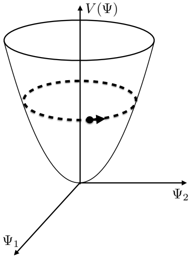
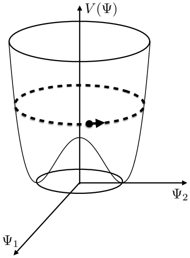
- Homogeneous cosmology.
- Linear perturbations.
- Deeply inhomogeneous level—near caustic formation.

Homogeneous cosmology





Choose a minimal energy configuration with $Q = \lambda^2 \dot{\varphi} \neq 0$

$\lambda \approx$ const

FRLW background: $\dot{\varphi} \gg H$

Perturbations

In $P(X)$-theories: $\omega^2 = c_s^2 \cdot \frac{k^2}{a^2}$

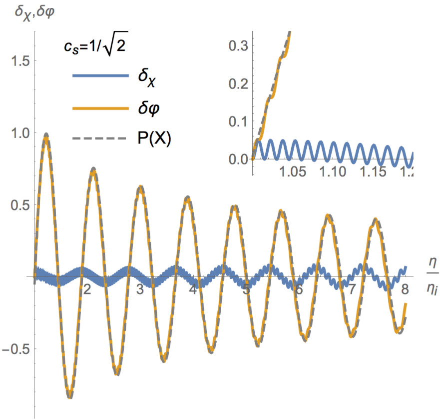
In complex scalar picture: 2 degrees of freedom $\Longrightarrow$

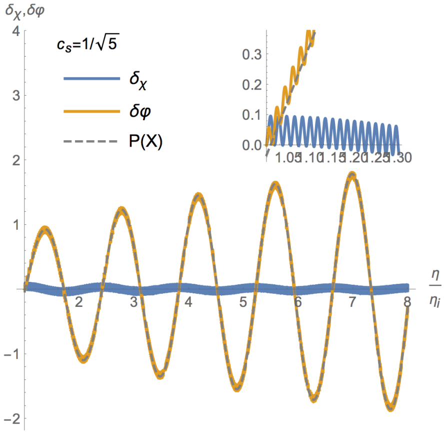
$$\boxed{\frac{k^2}{a^2} \ll \dot{\varphi}^2} \qquad \omega_1^2 = c_s^2 \cdot \frac{k^2}{a^2} + \mathcal{O}\left(\frac{k^4}{a^4 \dot{\varphi}^2}\right)$$

$$\boxed{M_{gap}^2 \simeq \dot{\varphi}^2} \qquad \omega_2^2 = M_{gap}^2 + \mathcal{O}\left(\frac{k^2}{a^2}\right)$$

$$\omega_2^2 \gg \omega_1^2$$

Modes propagation in the expanding Universe





Sub-luminal vs super-luminal case

$$\boxed{M_{gap}^2 = \frac{4\dot{\varphi}^2}{1 - c_s^2}}$$

For $c_s^2 < 1$ (sub-luminal case), then $M_{gap}^2 > 0$

If $c_s^2 > 1$ (super-luminal case), then $M_{gap}^2 < 0$

Super-luminality implies tachyon instabilities!

Adams et *al*'2006

Gradient unstable $P(X)$

$$\omega^2 = -|c_s^2| \cdot \frac{k^2}{a^2} \qquad |c_s| \ll 1 \quad \text{Also ghosts!!!}$$

In the complex field picture:

$$\omega_1^2 = -|c_s^2| \cdot \frac{k^2}{a^2} + \boxed{\frac{(1+|c_s^2|)^3}{4\dot{\varphi}^2} \cdot \frac{k^4}{a^4}}$$

Critical momentum, which cuts off the instability

$$\boxed{\frac{k_s}{a} \simeq |c_s|\dot{\varphi}} \Longrightarrow \Gamma = |c_s^2|\dot{\varphi} \qquad \Gamma \ll H_0$$

Inhomogeneous regime

Simplified example with pressureless perfect fluid: $\mathcal{L} = \frac{\lambda^2}{2}X - \frac{M^2\lambda^2}{2}$

This is not $P(X)$, but $V(\lambda) \to \frac{M^{2-\epsilon}\lambda^{2+\epsilon}}{2} \Longrightarrow P(X) \propto X^{1/\epsilon}$

Used to seed supermassive black holes, Sawicki et *al*'13

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Recall that in the complete model, one has

$$\mathcal{L} = \frac{1}{2}\left(|\partial_\mu \Psi|^2 - M^2 |\Psi|^2\right)$$

Manifestly free of caustic singularities!

Quantum pressure

$$\frac{\partial \lambda^2}{\partial t} + \nabla \left(\lambda^2 \mathrm{v}\right) = 0$$

$$\frac{\partial \mathrm{v}}{\partial t} + (\mathrm{v} \cdot \nabla)\mathrm{v} = \frac{1}{2M} \nabla \left(\frac{\Delta \lambda}{\lambda} \right)$$

Quantum pressure

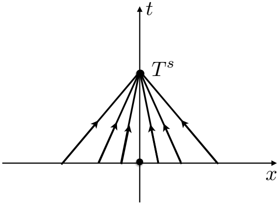
Madelung'1925

Widrow and Kaiser'93 Uhlemann, Kopp, and Haugg'14

Typical for scalar field Dark Matter (e.g., axions)

Resolving caustic singularity

$$v(t=0) \propto -x \Longrightarrow v = -\frac{x}{T^s - t}.$$



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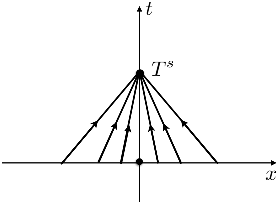

$$\boxed{T^s \to T^s - i\tau} \qquad \tau \propto \frac{1}{M} \qquad v = -\frac{x(T^s - t)}{(T^s - t)^2 + \tau^2}.$$

Conclusions

- $P(X)$-fluid and pressureless perfect fluid belong to the same class of models with two scalar fields. Both develop caustic singularities.
- Simple caustic free completion by means of the complex scalar.
- Completion works for subluminal $P(X)$ and $P(X)$ with gradient instabilities. For superluminal $P(X)$ the second degree of freedom is tachyonic.
- Mechanism of resolving caustic singularities: the collapse time is promoted to the complex number in the complete picture.

Thanks for your attention!